



ANALYTICAL METHOD OF REASONING FOR MATHEMATICAL TASKS AND GAMES SOLVING

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ABSTRACT

The purpose of the research is to review the analytical method of reasoning in the process of solving of mathematical logic tasks and games.

The research tasks include:

- Analysis of theoretical concepts related to different methodologies for solving of mathematical tasks;
- Introduction of fundamental theoretical concepts related to the analytical methodology of reasoning;
- Presentation of task for which the analytical methodology of reasoning is applied.

The report reflects the point of view of the researchers Mr. Portev and Mr. Nikolov regarding the two groups of methodologies for solving of mathematical tasks – the general logic approach and the specific mathematical approach.

The research presents in greater details the general logic methodologies for task solving that reflect the analysis and the synthesis in the process of seeking the solution of the task, and namely:

1. Analytical methodology of reasoning and its two variations – the ascending (or perfect analysis, diagram of Papp) and the descending (or imperfect analysis, diagram of Euclid);
2. Synthetic methodology of reasoning and analytic-synthetic methodology of reasoning (reasoning at both ends). This methodology is a combination between the synthetic methodology and one of the analytical methodologies of reasoning.

The advantages of the analytical methodology are very important. The following specifics of this methodology that lead to effective task solving are presented in the research:

-The analytical diagrams provide high repeatability and have control function in comparison with other diagrams. Even more, it seems that they have primary position to the others thus having a generating function. Something else which is of extreme importance is that after the conscious absorption of the knowledge, the control over the diagrams is performed with high degree of awareness.

-The diagrams for analysis and synthesis are anteriorly connected mostly with knowledge regarding the general logic methodologies and the heuristic rules. They are sooner strategies for seeking of task solutions and are transversal through different ages and activities.

-The diagrams are suitable to determine the stages of the solution of the mathematical task – to determine its components and to synthesize the information related to the main terminology that has got a descriptive and control function towards the rest of the components of the task. This leads to operability and heuristics of the analytical methodology of reasoning.

-The logical foundation of the analytical methodology and the high level of verbalization of its applicable diagrams show that the students can take part in the process of methodology building up.

Key words: analytical method, mathematical logic tasks, analysis, synthesize

The analytical methodology of reasoning is traditionally applied for solving of planimetry (or plane geometry) tasks. However, it was also found out that this approach can be successfully used for solving of some imaginative tasks of non-mathematical nature.

It is a well-known fact that the words “analysis” and “synthesis” are of Greek origin and that the ancient Greeks related them to the two of the basic methodologies of cognition.

To realize their effect on the system of education is a long process and is mostly related to the development of the heuristic abilities of the students.

S. L. Ribinstein describes the task solving process as a process of task “restatement” with constant analysis of the task condition and goal through the synthetic act of juxtaposition between the two.

The author believes that the main form of analysis is analysis through synthesis. “During

the process of reasoning the object gets consequently included in new relations thus demonstrating new qualities.” (1).

The researchers are united around the understanding that in the process of searching of the right approach for solving of a particular task, the analysis actually represents separation of the whole into its constituents or when the reasoning is directed from the task goal to the task conditions (2).

Synthesis is when the constituents get united in a whole or when the reasoning is directed from the task condition to the task goal.

In (3) the methods for tasks solving can be divided into two main groups: general logic methods and specific mathematical methods. The general logic methods are sooner reasoning methods for finding a solution from the correspondent task condition. These methods are generally valid for all sciences. The following methods that reflect analysis and synthesis can be classified as general logic methods: synthetic method of reasoning and analytical method of reasoning.

The description of these two methods retained their verbal characteristics until the middle of XX-th century. In the 1960s I. Ganchev offered a model of the three main types of mathematical tasks using implications and their solutions were presented through correspondent chains of implications. Thus, the analysis and synthesis in the search for solutions is justified logically.

The success in effective detection of implications strongly depends on the knowledge of definitions, axioms and theorems, on which those implications are based (4). The synthesis of knowledge in the area of the basic mathematical definitions increases the operative and the heuristic abilities of the students. This would have synergetic effect if a stress is put on definitions with explanatory and regulatory function towards the rest of the definitions.

Generally speaking, the interconnection between the reasoning methods and the effective task resolving can be found in ratiocination that include terminology combinations as: discovery and arrangement of implications; change of the goal of the activities; operative and heuristic character of the main definitions leading to higher activity in the process of task solving; engagement of the students in their creation; operative structures having explanatory and regulatory functions towards other structures; degree of awareness, verbalization and conscious control of the latter.

The following general logic task solving methods that reflect analysis and synthesis in the process can be specified:

1. Analytical method of reasoning – a methodology where the reasoning is directed from the task goal to the task condition. There are two different variations:
 - Ascending or perfect analysis (Diagram of Papp). The Diagram of Pap requires reasoning that shall be followed by another reasoning subject to proving.
 - Descending or imperfect analysis (Diagram of Euclid). The Diagram of Euclid starts with reasoning that needs to be proved. After that another reasoning ensuing from the first one shall be chosen.
2. Synthetic method of reasoning – methodology where the reasoning is directed from the task condition to the task goal.
3. Analytic – synthetic reasoning (reasoning from both ends). This methodology is a combination between the synthetic method and one of the analytic methods of reasoning.

In an attempt to provide more precise description of the above methods, the terminology “moving backwards strategy” and “moving forward strategy” is used in the specialized literature.

The opinion that the teacher hampers his students when he applies the synthetic method to search for a solution of mathematical task prevails. Normally, he starts from the point when he has seen the synthetic exposition and more precisely its beginning, during the initial task solving done by him before the classes.

The explanations can be presented in the following manner: as it is, then the result is; it is easy to notice that, which leads to the conclusion that; having in mind the fact that, the final result is

This can be explained with the fact that the task solving process is very complicated and individual and each description of this process would be relatively poor and ambiguous. Many researchers believe that the attempts to describe the process of task solution discovery simply represent an unnecessary explanations (sometimes the unsuitable questions and directions that do not make sense could have negative effect). Even those students who can ably solve mathematical tasks do not understand completely the difficulties that the students face in task solving process.

They remember whole clusters of mathematical tasks and their structure. In big extent they remember the components that represent part of the task solution.

In order to solve a task the students must carefully study the task condition and identify those parts of it that bring important information needed for the purposes of the task solution.

The students are required to demonstrate strong reflection in the process but this cannot happen without dedicated training.

If the two diagrams of the analytic method of reasoning are applied, does it facilitate the independent and the purposeful mathematical task solving?

The answer of this question is:

Both versions of the analytic method of reasoning compare two consequent ratiocination. This makes it easier for the student to set a goal and to achieve it.

When the diagram of the synthetic method is applied, usually we digress from what we want to prove. Consequently the unprepared mind of the student not only experiences difficulties to find the right direction but even when the starting point is shown by the teacher, he cannot accept it as appropriate and natural.

As a result the student is puzzled – he cannot understand how the one who solved the task managed to do this? In cases of complex mathematical task solution the student cannot make all the ratiocination needed to convince himself in the advisability of the chosen solution even after seeing it. If the teacher just reports those ratiocinations without demonstrating their advisability, this will not help student's creativity.

Unfortunately, most of the textbooks present task solutions through the synthetic method and do not offer enough support to those who use them. As mentioned above the students actually must disintegrate the task condition and identify all the parts that bring important information. Some of the textbooks contain directions for task solving but they are aimed rather to the facts and not towards application of mental techniques. Exception of this rule are the tasks which contain partial solution and require its completion as well as the tasks offering multiple options from which the student must choose the right one.

One of the most used techniques for selection of mathematical tasks is to sort them out as tasks–components for solution of particular task. This way the difficulty is reduced but the chance to apply the analytic method (even indirectly) towards the main task is lost. In this case the teacher must decide which task-components to be formed and to what degree they would allow the application of the analytic method of reasoning.

Using the analytic method of reasoning is closely related to the question of selection and sorting out of the mathematical tasks which would help the students to identify different implications constituting the decisions of the various tasks.

The above reasoning demonstrates that the Diagrams of Papp and Euclid used for the purposes of task solution analysis are of great heuristic value and can be considered as general strategies for searching of mathematical task solutions. The imperfect analysis is based on the rule that if “q” results from “p” and “q” is true then there is big possibility that “p” is true. The heuristic science accepts that a statement is true if a consequent statement is true.

In the light of this subject we are searching possibilities for purposeful application of the analytic method in task solving in the wide context of the latter.

Task 1. Two kids are plying the following game: from a sweets box containing 13 sweets each one of the kids takes in turn 1, 2 or 3 sweets. The winner is the one who takes the last sweet. Which one of the kids will always win and how?

Finding a solution

As we have no understanding of a winning overall strategy we try to consider the situation at the end of the game. We ask the question “How many sweets we would leave (on our turn before the last) to the other player to make sure that we will win the game?” It is enough to leave 4 sweets as 1, 2 or 3 sweets will remain after the last turn of the other player. The rules will allow us to take them and thus to win the game. So we might want to try the strategy “taking by fours”. We start first and leave successively 12, 8, 4 sweets in several steps. First we take one sweet. 12 sweets remain. The other player takes 1, 2 or 3 sweets. Remain 11, 10 or 9 sweets. We take 3, 2 or 1 sweet and make sure that 8 remain. The other player takes 1, 2 or 3 sweets and 7, 6 or 5 remain. We take 3, 2 or 1 and make sure that only 4 remain. The other player takes 1, 2 or 3 sweets. 3, 2 or 1 sweets remain. We take all of them including the last one. We win.

Task 2. One man had 12 liter of wine. He wanted to take half of it but didn't have a 6 liter vessel. He had two empty vessels – one of 8 liter and one of 5 liter. How could the man measure 6 liters through pouring the wine from vessel to vessel?

This task was included in the experimental program for education in mathematics for 4-grade pupil as well as in the program of University students. Very rarely someone of

them managed to find the correct answer unless they have already been solving its component-tasks or analogic tasks with simple solutions.

The solution of this task is presented below using the analysis from the Diagram of Euclid. We make the following drawing:

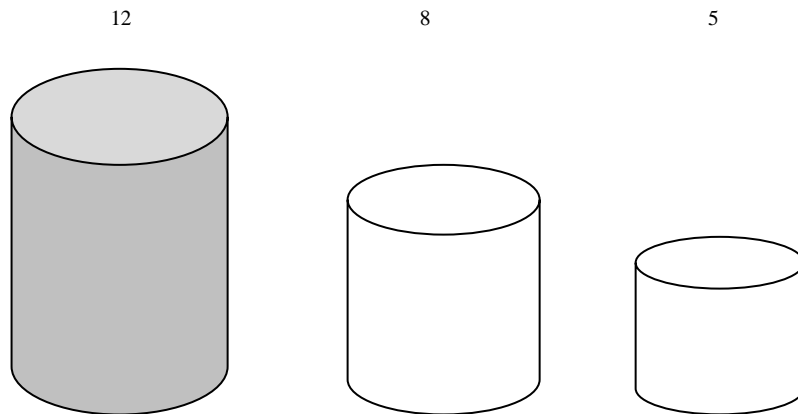


Figure 1.

We start pouring the liquid into the three vessels in such a way that the sum of the liters is always 12. We make several tests:

12	0	0
4	8	0
0	8	4
.....		
.....		

The test we do are not based on certain logic, there is no strategy regarding the intermediate result we want to achieve and we rely on the chance.

Let us try another approach admitting that separating the liquid into two equal portions is possible. Other words, we are looking for a way to pour the liquid in shares following the distribution 6 6 0. From 6 6 0 (let us call it a First Row distribution) we receive four distributions of Second Row.: 4 8 0, 1 6 5, 12 0 0, 6 1 5. Now we have to answer the question which one of these distributions can be returned back to the First Row, i.e. we can talk about reversible distributions.

The first and the third distribution cannot be returned back into initial position whereas the second and the forth distribution can. Using analogic reasoning, we discover in distribution 1 6 5 the reversible distribution 1 8 3 which is the key to the task solving. Let us use the synthetic approach:

I	12	0	0
II	4	8	0
III	4	3	5
IV	9	3	0
V	9	0	3
VI	1	8	3
	1	6	5
	6	6	0

It is important to reach to the sixth distribution which is reversible. After doing this, it will be easy to separate the liquid equally into two different vessels.

In this example the sequence in the use of the synthetic and the analytic method of reasoning can be seen.

Task 3. From the basket with walnuts Pippy took half of them plus one. Anika took half of the remaining walnuts plus another 2. In the end, Mister Nilsson took half of the remaining walnuts plus the last 3. How many walnuts were there initially in the basket and how many walnuts everyone took?

Solution 1 (arithmetical solution)

In the end Mister Nilsson took half of the remaining walnuts as well as the last 3 walnuts. Therefore, the last 3 walnuts are the second half off the walnuts that remained for Mr. Nilsson, i.e. he took 6 walnuts in total. Anika then has left 6 walnuts which 2 walnuts less than the half that she took. Therefore, the half of remaining walnuts that Anita has taken consists of 8 walnuts. She has taken 10 walnuts in total. Pippy then has left 16 walnuts which represents half of the walnuts that she has taken minus 1. Therefore, the half that Pippy has taken consists of 17 walnuts. So initially there were 34 walnuts in the basket. Pippy has taken 18 walnuts, Anika – 10 walnuts and Mr. Nilsson – 6 walnuts.

Solution 2 (algebraic solution)

The task can be solved by making an equation of one variable with rational coefficients. If we denote with “y” the walnuts that remain for Annika, we get a simplified form of the equation

$$y - 0,5y - 2 - 0,5(y - 0,5y - 2) - 3 = 0$$

$$0,5y - 2 - 0,5(0,5y - 2) - 3 = 0$$

$$y - 4 - 0,5y + 2 - 6 = 0$$

$$0,5y = 8, \text{ т.е. } y = 16.$$

If we denote with “x” the number of walnuts that were initially in the basket, we will have the following equation:

$$x - 0,5x - 1 = 16$$

$$0,5x = 17, \text{ i.e. } x = 34.$$

Another idea for solving of this task that can be illustrated is to find a typical quality of the total number of walnuts that initially have been in the basket and to search for a solution through tests. Such option would be dividing the total number of walnuts on 4 with 2 walnuts always remaining. If there is a solution, it will be amongst the numbers 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, ...

An example for a component-task of this task will be the following:

Task 3*. Pippy has taken from the basket half of the walnuts plus 1. After that Mr. Nilsson has taken half of the remaining walnuts plus the last 3. How many walnuts were there in the basket initially and how many walnuts everyone took?

The following task has the same solution (in respect of strategy):

Task 3.** Pippy has taken from the basket half of the walnuts plus 1. After that Anika has taken half of the remaining walnuts plus another 2. In the end Mr. Nilsson has taken one third part of the remaining walnuts plus the last 4. How many walnuts were there initially in the basket and how many walnuts everyone has taken? (5)

According to us this mathematical task has the following developing functions:

- The task requires the primary school pupil to find the solution from the end of the task;
- The solution through equation requires correct choice of variables; the idea of substitution can be introduced;
- Allows to foresee qualities of the variable and to find a solution.

The illustrations of the analytical method of reasoning through logic tasks and games have three extremely important advantages:

- They don't have the burden of too much mathematical information. This allows the method to be seen in a pure form and to recognize the Socrates dialogues with "reasonable" questions;
- They can be used without consideration of the currently studied material in order to achieve greater repeatability of the method;

- They demonstrate the applicability of the mathematics and play the role of motivator-tasks as they are transversal through students' age and activities.

In conclusion the following specifics of the analytic method of reasoning that lead to effective mathematical tasks solving can be outlined:

1. The diagrams for analysis provide high repeatability of the method and have control function over the rest. Moreover, they seem to have primacy over the other, i.e. generating function. In addition, there is something of extreme importance - after the conscious learning the students have control over the diagrams with high degree of awareness.
2. The diagrams for analysis and synthesis are related as priority mostly with knowledge for general logic methods and heuristic rules. They are sooner strategies for searching of mathematical task solutions. Typically, they are transversal through different ages and activities.
3. Suitable disintegration of task solutions – components and information synthesis around the main terminology that has explanatory and control function towards the rest of the components leading to efficiency and heuristics of the analytical method of reasoning. The possibilities for selection of implications where the reasoning and the conclusions follow directly from one another provide purposefulness of the activities and active participation in the task solving process.
4. The logical exposition of the analytical method of reasoning and the high level of verbalization of its application diagrams show that the students can directly participate in the process of model construction.

The following research directions of the analytical method of reasoning can be outlined:

- Specifics related to inclusion of typical model of task solving with transversal nature;
- Specifics related to the ability of both students in different school grades and people in mature age to learn the analytical method;
- Specifics related to the ability of both students with difficulties at school and gifted students to learn the analytical method;

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